Discrete Mathematics 2025 Spring

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- 4.1 Definition and Representation of Relations
- 4.2 Operations on Relations
- 4.3 Properties of Relations
- **4.4 Equivalence Relations and Partial Order Relations**

- **Definition 4.18:** let *R* be a relation on a non-empty set. If *R* is reflexive, symmetric, and transitive, then *R* is called an ***equivalence relation*** on *A*. If *R* is an equivalence relation and $<x,y>\in R$, **we say that *x*** is equivalent to ***y*, denoted as** $x \sim y$.

e.g. **Example:** Verify that *R* is an equivalence relation on *A*.

Let $A=\{1, 2, \ldots, 8\}$, and define the relation ***R on A as follows*** :

$R=\{<x,y> \mid x,y\in A \wedge x\equiv y \pmod 3\}$

where $x\equiv y \pmod 3$ means that ***x and y*** are congruent modulo 3, i.e., the remainder when **x** is divided by **3** is equal to the remainder when ***y*** is divided by **3**.

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e.g. **Example: Let** $A=\{1, 2, \ldots, 8\}$, and define the relation *R on A as follows*: $R=\{\langle x,y\rangle \mid x,y\in A \wedge x\equiv y \pmod 3\}$

where $x\equiv y \pmod 3$ means that *x and y* are congruent modulo 3, i.e., the remainder when **x** is divided by **3** is equal to the remainder when ***y*** is divided by **3**.

It is easy to verify that *R* is an equivalence relation on *A*, because:

$\forall x\in A$, if $x\equiv x \pmod 3$ （reflexivity）

$\forall x,y\in A$, if $x\equiv y \pmod 3$, then $y\equiv x \pmod 3$ （symmetry）

$\forall x,y,z\in A$, if $x\equiv y \pmod 3$, $y\equiv z \pmod 3$, then $x\equiv z \pmod 3$ （transitivity）

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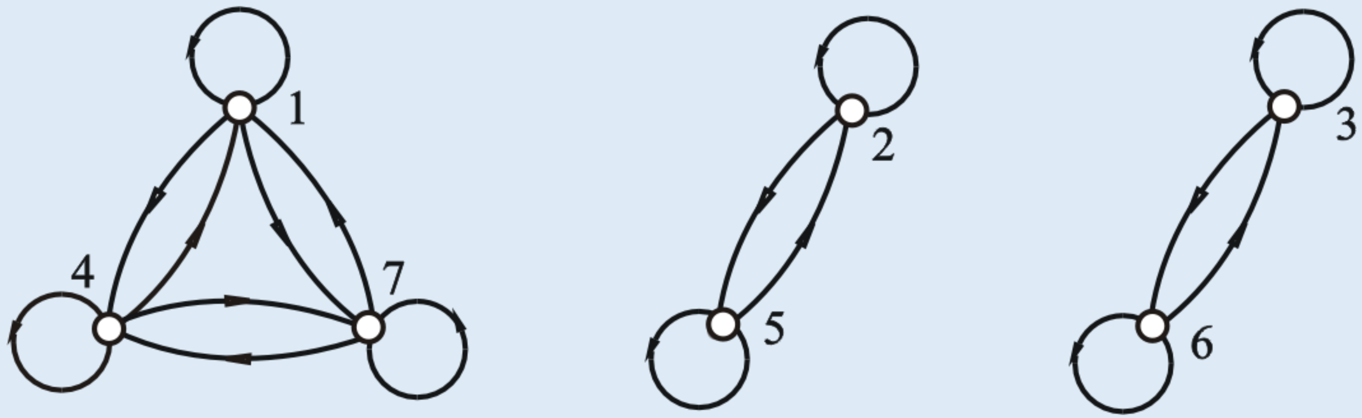
- Relation Graph of the Modulo 3 Equivalence Relation on A

Let $A=\{1, 2, \ldots, 7\}$,

$R=\{ \langle x,y \rangle \mid x,y \in A \wedge x \equiv y \pmod 3 \}$

The relation graph of *R* is shown below：



- **Definition 4.19: let** ***R*** be an equivalence relation on a non-empty set *A*, $\forall x \in A$, define $[x]_R = \{ y \mid y \in A \wedge xRy \}$
  We call $[x]_R$ the ***equivalence class*** of ***x under R***, or simply the equivalence class of ***x***, denoted as **[*x*].**

Note: $[x]_R$ is the set of all elements in *A* that are equivalent to **x** under the relation *R*.

$$[x]_R = \{y \in A \mid (x,y) \in R\}$$

e.g. **Example:** Equivalence classes of the modulo 3 equivalence relation on **A={1, 2, … , 8}：**

[1]=[4]=[7]={1,4,7}
[2]=[5]=[8]={2,5,8}
[3]=[6]={3,6}

- The three equivalence classes with remainders 1, 2, and 0 are disjoint, and their union is *A*.

- **Theorem 4.8:** Partition Theorem of Equivalence Classes.

Let R be an equivalence relation on a non-empty set A

The following conclusions hold:

(1) $\forall x \in A$, $[x]$ is a non-empty subset of A.

(2) $\forall x, y \in A$, if xRy, then $[x]=[y]$.

(3) $\forall x, y \in A$, if $x \not R y$, then $[x]$ and $[y]$ are disjoint.

(4) $\bigcup_{x \in A} [x] = A$, the union of all equivalence classes is equal to A.

↳ Proof of the Partition Theorem of Equivalence Classes

- **Theorem 4.8(1):** $\forall x \in A$, $[x]$ is a non-empty subset of A.

Proof: From the definition of equivalence classes, $\forall x \in A$, we have $[x] \subseteq A$. By reflexivity, xRx， so $x \in [x]$, which implies that $[x]$ is non-empty. Since all elements in the equivalence class $[x]$ are selected from the set A， it follows that $[x]$ is a subset of A.

- **Theorem 4.8(2):** $\forall x, y \in A$, if xRy, then $[x]=[y]$.

Proof: For any element a in $[x]$， $(x,a) \in R$. Since $(x,y) \in R$ and R is transitive, we can conclude that $(y,a) \in R$, so $a \in [y]$. This proves that $[x] \subseteq [y]$.

Similarly, we can prove that $[y] \subseteq [x]$. Therefore $[x]=[y]$.

- **Theorem 4.8(3):** $\forall x, y \in A$, **if** $x \not R y$, **then** $[x]$ and $[y]$ are disjoint.

Proof:

Suppose $[x] \cap [y] \neq \emptyset$, then there exists an element $z \in [x] \cap [y]$, which implies $z \in [x] \wedge z \in [y]$, **then** $\langle x,z \rangle \in R \wedge \langle y,z \rangle \in R$ ***holds***.

By the **transitivity** and **symmetry** of R, ***then*** $\langle x,y \rangle \in R$,

contradicted to $x \not R y$.

- **Theorem 4.8(4):**

  $\bigcup_{x\in A}[x] = A$, the union of all equivalence classes is A.

- **Proof1:** $\cup_{x\in A}[x] \in A$, For any y.

  $y \in \cup_{x\in A}[x] \Leftrightarrow \exists x\ (x\in A \wedge y\in[x])$

  $\Rightarrow y\in[x] \wedge [x]\subseteq A \Rightarrow y\in A$

  $Then\ \cup_{x\in A}[x] \in A)$

- **Proof2:** $A \in \cup_{x\in A}[x]$, For any y.

  $y\in A \Rightarrow y\in[y] \wedge y\in A \Rightarrow y\in \cup_{x\in A}[x]$

  Hence $A \subseteq \cup_{x\in A}[x]$ host.

- **Thus, we conclude that:** $\cup_{x\in A}[x] = A$.

- **Definition 4.20:** Let R be an equivalence relation on a non-empty set A. The set of equivalence classes of R is called the ***quotient set*** of A with respect to R, denoted by A/R,

$$A/R = \{ [x]_R \mid x \in A \}$$

e.g. »» **Example:** *Let* $A=\{1, 2, \ldots, 8\}$, the quotient set of A with respect to the equivalence relation R modulo 3 is: $A/R = \{ \{1, 4,7\}, \{2, 5, 8\}, \{3, 6\} \}$

- The quotient sets of A with respect to the identity relation and the universal relation are:

$$A/I_A = \{ \{1\},\{2\}, \ldots ,\{8\}\}$$

$$A/E_A = \{ \{1, 2, \ldots ,8\} \}$$

- 4.4.1 Equivalence Relations
- 4.4.2 Equivalence Classes and Quotient Sets
- 4.4.3 Partition of a Set
- 4.4.4 Partial Order Relations
- 4.4.5 Partially Ordered Sets and Hasse Diagrams

- **Definition 4.21:** Let *A* be a non-empty set, *A* family of subset π ($\pi \subseteq P(A)$), if it satisfies the following conditions:

  (1) $\varnothing \notin \pi$; (2) $\forall x \forall y\,(x,y \in \pi \wedge x \neq y \rightarrow x \cap y = \varnothing)$; (3) $\cup \pi = A$

  Then π is called a ***partition of A***, and the element of π are called blocks of the partition of *A*.

e.g. **Example:** Let $A = \{a, b, c, d\}$, given the partitions:

$\pi_1 = \{\{a, b, c\}, \{d\}\}$, $\pi_2 = \{\{a, b\}, \{c\}, \{d\}\}$

$\pi_3 = \{\{a\}, \{a, b, c, d\}\}$, $\pi_4 = \{\{a, b\}, \{c\}\}$

$\pi_5 = \{\varnothing, \{a, b\}, \{c, d\}\}$, $\pi_6 = \{\{a, \{a\}\}, \{b, c, d\}\}$

Then π_1, π_2 *are partitions of A*, while the others are not.

- The quotient set *A/R is a partition of A* .
- Different quotient sets correspond to different partitions.
- Given any partition π of A , we define a relation ***R on A*** as follows:

$$R = \{<x,y> \mid x,y \in A \text{ and } [x]_\pi = [y]_\pi\}$$

($[x]_\pi = [y]_\pi$: ***x*** and ***y*** are in the same partition block of ***π***)

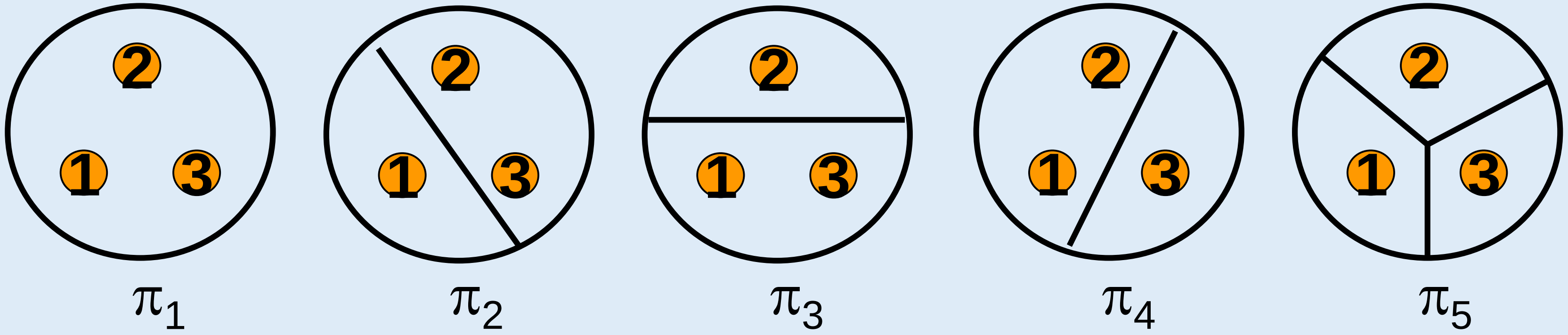
Then, ***R*** is an equivalence relation on ***A***, and the quotient set determined by this equivalence relation is exactly π.

e.g. **Example:** List all equivalence relations on $A = \{1,2,3\}$

Solution approach:

First, determine all partitions of *A*, and then write out the corresponding equivalence relations based on these partitions.

Determine all partitions of A={1,2,3}



π_1 corresponds to the universal relation E_A

π_5 corresponds to the identity relation I_A

π_2,π_3 $\pi_3$correspond to the equivalence relations R_2, R_3 和 R_4.

$R_2=\{<2,3>,<3,2>\}\cup I_A$

$R_3=\{<1,3>,<3,1>\}\cup I_A$

$R_4=\{<1,2>,<2,1>\}\cup I_A$

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e.g. **Example:** Let $A=\{1,2,3,4\}$

- Define a binary relation R on $A\times A$:

  $<<x, y>,<u, v>>\in R \Leftrightarrow x+y = u+v$, find the partition induced by R.

  Solution: $A\times A=\{<1,1>, <1,2>, <1,3>, <1,4>, <2,1>, <2,2>, <2,3>, <2,4>,<3,1>, <3,2>, <3,3>, <3,4>, <4,1>, <4,2>,<4,3>, <4,4>\}$

- According to the sum condition $<x,y>$ and $x+y=2,3,4,5,6,7,8$ which partition $A\times A$ into 7 equivalence classes:

$$(A\times A)/R=\{\{<1,1>\}, \{<1,2>,<2,1>\}, \{<1,3>, <2,2>, <3,1>\}, \{<1,4>, <2,3>, <3,2>, <4,1>\}, \{<2,4>, <3,3>, <4,2>\}, \{<3,4>, <4,3>\}, \{<4,4>\}\}$$

Definition 4.22:

A relation on a non-empty set that is **reflexive, antisymmetric, and transitive** is called a ***partial order relation*** on *A*, denoted ***by*** **≼. If <*x*, *y*>∈≼,** then we write it as ***x*≼*y*,** which is read as ***x*** "less than or equal" ***y*.**

e.g. Examples:

- The **identity relation** I_A **on set** *A* is a partial order relation on *A* **.**
- The **less than or equal to** relation, **divisibility** relation, and **subset inclusion** relation are also partial order relations on their respective sets.

Definition 4.23: Comparability

Let *R* be a **partial order relation** on a non-empty set *A*,

- **For *x*, *y*∈*A***, we say that ***x and y*** are ***comparable*** if and only if ***x*≼*y* or *y*≼*x***.
- **Condition for incomparability:** If no partial order relation relates ***x*** and ***y***, then they are **not comparable**.

Definition 4.24: Total Order

- If *R* is a partial order on a non-empty set *A*, ∀***x*, *y*∈*A*, *x and y*** are always comparable, ***R*** is called a ***total order***.

e.g. **Examples:**

- The "**less than or equal to**" relation on numerical sets (such as real numbers and integers) is a **total order**.'
- The **divisibility relation** is **not a total order** on the set of positive integers.

■ **Definition 4.25:** Covering

$x,y \in A$, if $x \prec y$ and there is no $z \in A$ *such that* $x \prec z \prec y$, then we say that ***y covers x***.

- **Such as:** On the set {1, 2, 4, 6} with the divisibility relation:

  2 covers1, 4 and 6 covers 2.

  4 doesn't cover 1.

4.4 Equivalence Relations and Partial Order Relations

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- **Definition 4.26： Partially ordered set.**

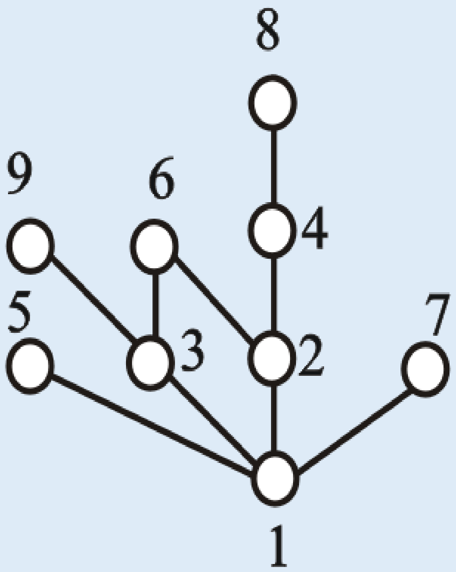
  A ***partially ordered set*** **(poset)** consists of a set *A* together with a partial order relation ≼ , denoted as **<*A*,≼>.**

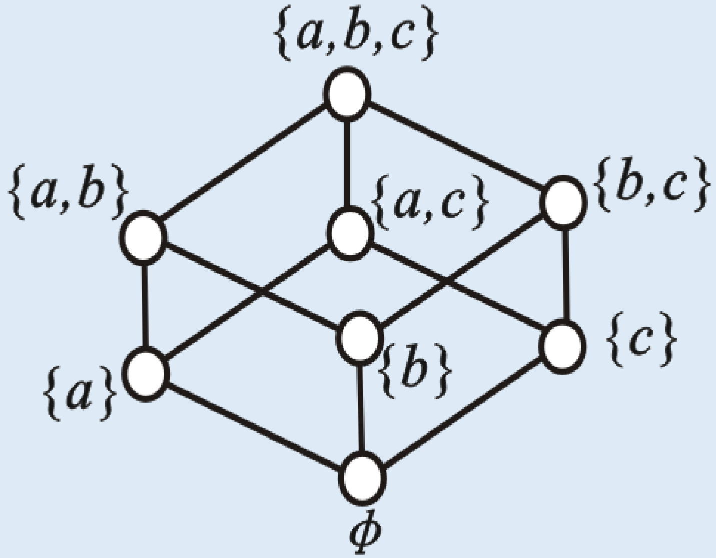
- **Such as：**
  - The set of integers with the "less than or equal to" relation forms a poset **<Z,≤>.**
  - The power set *P*(*A*) with the subset inclusion relation forms a poset $<P(A),R_{\subseteq}>$.

- **Hesse Diagram:** A simplified graphical representation of a partial order that eliminates reflexivity, antisymmetry, and transitivity in the diagram.
- **Characteristics:**
  - Each node has no self-loops.
  - The order between two connected nodes is represented by their relative position: **Lower-positioned elements come earlier in the order**.
  - There is an edge between two nodes **if and only if** they have a covering relation.
- A Hasse diagram is a special type of relational graph for posets, with transitive edges removed and implicit direction.

e.g. Example : $<\{1, 2, 3, 4, 5, 6, 7, 8, 9\}, R_{Div}>$

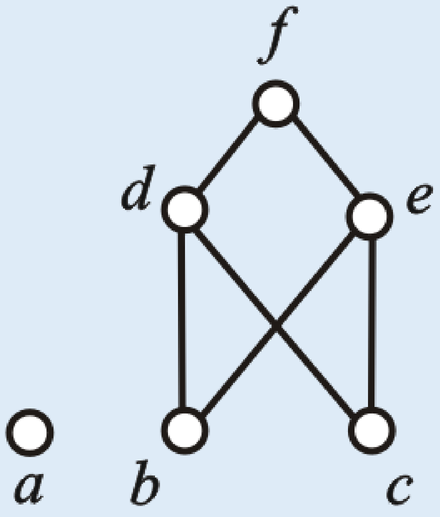
$<P(\{a, b, c\}), R_{\subseteq}>$





e.g.⋙**Example :** Given the Hasse diagram of the **partially ordered set <*A*,*R*>** shown below, find the expression for the **set *A* and the relation *R*.**

- **Solution:** ①Identify the Elements in *A*;②Extract the Covering Relations;③Complete transitive and reflexive relations.



$A=\{a, b, c, d, e, f\}$

$R=\{\langle b,d\rangle,\langle b,e\rangle,\langle b,f\rangle,\langle c,d\rangle,\langle c,e\rangle,\langle c,f\rangle,\langle d,f\rangle,\langle e,f\rangle\}\cup I_A$

↳ Key elements of a poset and their properties

- **Definition 4.27:** Let $<A,\preccurlyeq>$ be a **partially ordered set (poset)**, $B \subseteq A$, $y \in B$.

(1) if $\forall x(x \in B \rightarrow y \preccurlyeq x)$, then y is called the ***least element*** of B.

(2) if $\forall x(x \in B \rightarrow x \preccurlyeq y)$, then y is called the ***greatest element*** of B.

(3) If $\forall x(x \in B \wedge x \preccurlyeq y \rightarrow x = y)$, then y is called a ***minimal element*** of B.

(4) If $\forall x(x \in B \wedge y \preccurlyeq x \rightarrow x = y)$, then y is called a ***maximal element of*** B.

- **Properties:**
  - In a **finite set**, minimal and maximal elements **always exist** and **may not be unique.**
  - **Least and greatest elements are not guaranteed to exist**, but if they do, they are unique.
  - The **least element** is always a **minimal element.**
  - The **greatest element** is always a **maximal element.**
  - **Isolated nodes** are **both minimal and maximal elements.**

- **Definition 4.28:** **Let** $\langle A, \preccurlyeq \rangle$ be a **partially ordered set (poset)**, and let $B \subseteq A$, $y \in A$.

(1) **If** $\forall x\ (x \in B \rightarrow x \preccurlyeq y)$, then y is called an **upper bound** of B.

(2) **If** $\forall x\ (x \in B \rightarrow y \preccurlyeq x)$, then y is called a **lower bound** of B.

(3) **Let** $C = \{y \mid y$ is an upper bound of $B\}$, the **least element** of C, if it exists, is called the **least upper bound (supremum) of** B or the **supremum**.

(4) **Let** $D = \{y \mid y$ is a lower bound of $B\}$, The **greatest element** of D, if it exists, is called the **greatest lower bound (infimum) of** B or the **infimum**.

Properties:

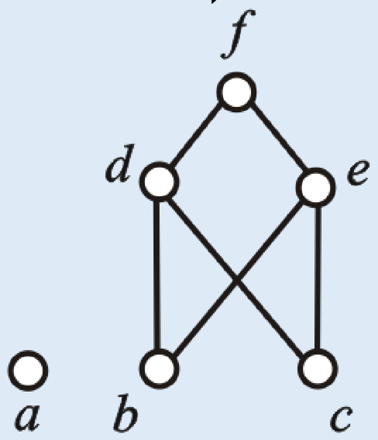
- Lower bounds, upper bounds, infimum, and supremum are **not always guaranteed to exist.**
- Lower bounds and upper bounds, if they exist, **may not be unique**.
- The infimum and supremum, if they exist, are unique.
- The least element of a set is its infimum, and the greatest element is its supremum, however, the **reverse is not always true**.

↳ Upper/lower bounds, supremum, infimum of a poset（e.g.）

e.g.»» **Example:**

(1) Given the **partially ordered set** $<A, \preccurlyeq>$ as shown in the diagram, find the minimal elements, least element, maximal elements, and greatest element of A.

(2) Let $B = \{b, c, d\}$, find the **lower bounds**, **upper bounds**, **infimum (greatest lower bound)**, and **supremum (least upper bound)** of B.



Solution (1)： Minimal elements: a, b, c;

Maximal elements： a， f;

No least element or greatest element.

Solution (2) :

Lower bounds and greatest lower bound do not exist.

Upper bounds: d, f

Least upper bound (supremum): d

↳ Special Subsets of a Poset: Chains and Antichains

- **Definition 4.29:** Let <*A*,≼>be a **partially ordered set (poset)**, and *B*⊆*A*.

(1) If for all ∀***x***,***y***∈***B***, ***x*** and ***y*** are **comparable**, then ***B*** is called a ***chain*** in *A*, The number of elements in ***B*** is called the **length of the chain.**

(2) If for all ∀***x***,***y***∈***B***, ***x***≠***y***, ***x*** and ***y*** are **not comparable**, then ***B*** is called an ***antichain*** in *A*, The number of elements in B is called the **length of the antichain..**

e.g.>>>**Examples:** In the poset <{1,2,...,9},|>, **{1,2,4,8}** is a **chain** of length **4**, **{1,4}** is a **chain** of length **2**, **{2,3}** is an **antichain** of length **2**. The **singleton set** {2 has length **1** and is **both a chain and an antichain.**

↳ Antichain Decomposition Algorithm for Poset

- **Theorem 4.9**: Let <*A*,≼> be a **partially ordered set (poset)**. If the length of the longest chain in *A* is ***n***, then the poset can be decomposed into ***n disjoint antichains***.
- **Algorithm 4.2: Antichain Decomposition Algorithm for Posets.**

  Input: A partially ordered set *A*

  Output: Antichains B_1, B_2, …

  1. $i \leftarrow 1$
  2. $B_i \leftarrow$ the set of all **maximal elements** in A (which is an antichain)
  3. $A \leftarrow A - B_i$
  4. if $A \neq \varnothing$
  5. $i \leftarrow i+1$
  6.   Go to 2

- **Topological Sorting:** Expanding a **partially ordered set (poset)** into a **totally ordered set** is called ***topological sorting***.
- **Algorithm 4.3: Topological Sorting**

Input: A partially ordered set A

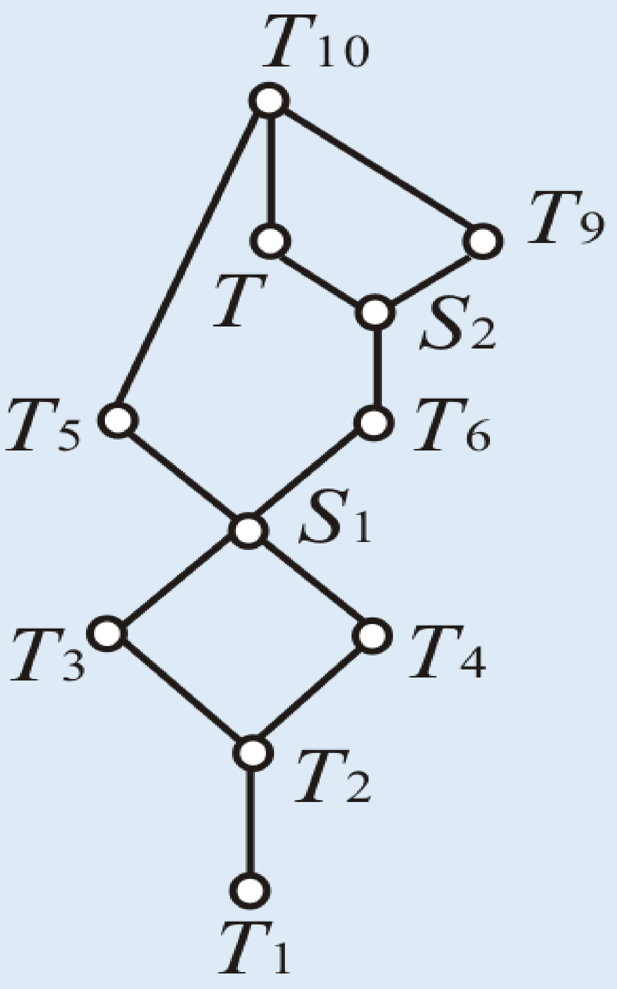
Output: A sorted order of elements in A

1. $i \leftarrow 1$
2. Select a **minimal element** a_i from A and consider it the **smallest element**
3. $A \leftarrow A - \{a_i\}$
4. if $A \neq \varnothing$
5. $\quad i \leftarrow i+1$
6. $\quad$ Go to 2

↳ Topological Sorting - Extending a Poset to a Total Order(e.g.)

e.g. **Examle**: A set of tasks *A* is given with partial order constraints, and its Hasse diagram is shown in the figure.

$A = \{T_1, T_2, T_3, T_4, T_5, S_1, T_6, S_2, T, T_9, T_{10}\}$



Check whether the following topological order is valid.

such as:

$T_1, T_2, T_3, T_4, S_1, T_5, T_6, S_2, T, T_9, T_{10};$

$T_1, T_2, T_3, T_4, S_1, T_6, S_2, T, T_9, T_5, T_{10};$

Objective：

Key Concepts：